Metric Einstein summation

SQUARED MAHALANOBIS DISTANCE

$$d^2 = q^T G^{-1} q$$

COMPLETE EXAMPLE

One residual in two coordinate systems

```
%%%
One sensor residual in two coordinate systems.
%%%
flatppl_compat = "0.1"
r = [2.0, 1.0]
I2 = eye(2)
J = rowstack([[1.0, 1.0], [0.0, 1.0]])
q = J * r
G = J * transpose(J)

I2: old[] := r[.i^] * r[.i_]
G: dist2[] := q[.i^] * q[.i_]
raw[] := q[.i] * q[.i]

dual = linsolve(G, q)
parts[.i] := q[.i] * dual[.i]

G: grad[.i_] := 2.0 * q[.i_]
step = [0.0, 1.0]
G: slope[] := grad[.i_] * step[.i^]

noise = MvNormal([0.0, 0.0], G)
log_L = logdensityof(noise, q)
outputs = record(distance2 = dist2,
    log_L = log_L, directional = slope)
```

The metric argument

Pass the covariance `G` as the upper metric g^{ij}. Within a `G:` contraction, the marks on each index determine which components FlatPPL reads.

Indexed expression	Matrix rows
`G[.i^, .j^]`	`[2, 1]`, `[1, 1]`
`G[.i_, .j_]`	`[1, -1]`, `[-1, 2]`
`G[.i^, .j_]`	`[1, 0]`, `[0, 1]`

The explicit contraction

```
d = metricsum(G, [], q[.i^] * q[.i_])
```

Empty brackets request a scalar. Each repeated contracted index must occur once upper and once lower. A free index retains the variance declared on the left.

ANNOTATED LINES

A Transform the covariance 9–10

```
q = J * r
G = J * transpose(J)
```

The original residual is `r = [2, 1]` with identity noise covariance. The instrument reports `q = [x + y, y]`. Its covariance is $G = JJ^T$, with rows `[2, 1]` and `[1, 1]`.

B Contract the two factors 13

```
G: dist2[] := q[.i^] * q[.i_]
```

Components $\underset{3 \times 2 + 1 \times (-1) = 6 - 1 = 5}{\texttt{[3, 1]} \quad \texttt{[2, -1]}}$

`.i^` reads the stored vector q. `.i_` reads $G^{-1}q$. The repeated index pairs the components. Their sum of products gives the original squared distance. Line 14 squares the stored entries and gives 10.

C Check the lower components 16–17

```
dual = linsolve(G, q)
parts[.i] := q[.i] * dual[.i]
```

`linsolve` solves $2u_1 + u_2 = 3$ and $u_1 + u_2 = 1$. Subtract the equations to obtain $u_1 = 2$, then $u_2 = -1$. Line 17 leaves `.i` free, so `parts` keeps both products `[6, -1]`.

D Differentiate the squared distance 19–21

```
G: grad[.i_] := 2.0 * q[.i_]
step = [0.0, 1.0]
G: slope[] := grad[.i_] * step[.i^]
```

In lower components the derivative is $2G^{-1}q$, or `[4, -2]`. FlatPPL raises the lower output for storage as `[6, 2]`. Reading `grad[.i_]` recovers the derivative. A step along the second axis has slope −2.

Finite displacement

Hold the first reported component at 3. Increasing the second to $1 + h$ yields the original residual $(2 - h, 1 + h)$. Its squared length is $5 - 2h + 2h^2$, which verifies the derivative at $h = 0$.

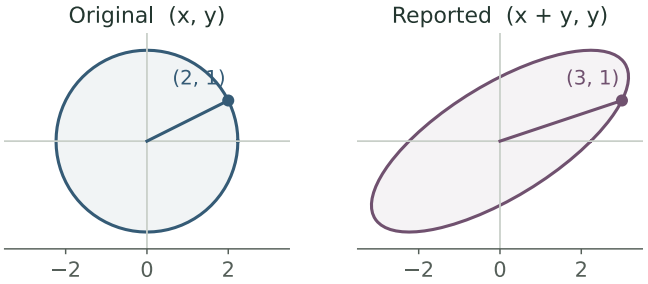
Norm of the derivative covector

```
G: norm2[] := grad[.i_] * grad[.i^]
```

With lower components $a = [4, -2]$, the squared norm is $a^T G a$. Multiplication by G gives the upper components $[6, 2]$, so the contraction is $24 - 4 = 20$.

COORDINATE AND LIKELIHOOD CHECKS

One contour in two coordinate systems



The same residual lies on the contour $d^2 = 5$ in both panels. The shear maps the circle to an ellipse. At fixed q_1, increasing q_2 increases y and decreases x by the same amount.

The Gaussian score

The two-dimensional log density is $-\frac{1}{2}(d^2 + \log\det G + 2\log(2\pi))$. Here $\det G = 1$, so `log_L` is −4.33788. For a general change of scale the determinant also contributes.

Change the units of both coordinates

```
q2 = 2 .* q
G2 = 4 .* G
G2: d2[] := q2[.i^] * q2[.i_]
```

The squared distance stays 5. Doubling both coordinates multiplies area by four. The log density therefore falls by $\log 4$.

An index left free

```
G: T[.i^, .j^] := q[.i^] * q[.j^]
G: w[.i^] := T[.i^, .j^] * q[.j_]
```

`T` contains the outer product. Contracting its second slot gives $w^i = q^i d^2$. In reported coordinates the result is `[15, 5]`.

Spacetime signature

```
eta = diagmat([1, -1, -1, -1])
p = [5, 3, 0, 0]
eta: m2[] := p[.a^] * p[.a_]
```

A metric may be indefinite. With signature $(+,-,-,-)$ and $c = 1$, this contraction gives the invariant mass squared $5^2 - 3^2 = 16$.

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