Two spins and a partition function

ONE BOND AT ZERO FIELD

$$Z = e^{K} + e^{-K} + e^{-K} + e^{K}$$

COMPLETE EXAMPLE

A finite Gibbs measure

```
%%%
Two Ising spins, one bond, zero applied field.
%%%
flatppl_compat = "0.1"
ratio = 3.0
K = log(ratio) / 2
same = exp(K)
opposite = exp(-K)
pp = [1, 1]
pm = [1, -1]
mp = [-1, 1]
mm = [-1, -1]
M = superpose(
    weighted(same, Dirac(pp)),
    weighted(opposite, Dirac(pm)),
    weighted(opposite, Dirac(mp)),
    weighted(same, Dirac(mm)))
Z = totalmass(M)
P = normalize(M)
p_pp = densityof(P, pp)
p_pm = densityof(P, pm)
p_mp = densityof(P, mp)
p_mm = densityof(P, mm)
p_aligned = p_pp + p_mm
correlation = p_aligned - (p_pm + p_mp)
Z_exact = 4 * cosh(K)
```

Spin correlation

`closed_form = tanh(K)`

The product $s_1 s_2$ takes the value +1 for aligned spins and −1 for opposed spins. Its expectation, `correlation`, is $\frac{3}{4} - \frac{1}{4} = \frac{1}{2}$ at this coupling.

Vary the temperature

The weight ratio is $r = \exp(2K)$. The probability of alignment is $\frac{r}{1+r}$. Change `ratio` to compare the three cases below.

Ratio	$\frac{1}{3}$	1	3
Aligned	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
Correlation	$-\frac{1}{2}$	0	$\frac{1}{2}$

For positive J, cooling increases K and favours alignment.

ANNOTATED LINES

A Bond energy and Boltzmann weight 8–9

```
same = exp(K)
opposite = exp(-K)
```

For one bond at zero field, write $K = \beta J$ and $E = -J s_1 s_2$. The Boltzmann weights are e^{K} for aligned spins and e^{-K} for opposed spins.

B Assemble the state space 15–19

```
M = superpose(
    weighted(same, Dirac(pp)),
    weighted(opposite, Dirac(pm)),
    weighted(opposite, Dirac(mp)),
    weighted(same, Dirac(mm)))
```

Place a unit `Dirac` mass at each ordered pair and multiply it by its Boltzmann weight with `weighted`. Add the four atoms with `superpose`. The pairs `[1, -1]` and `[-1, 1]` are distinct states.

C Partition function and normalization 21–22

```
Z = totalmass(M)
P = normalize(M)
```

Compute the partition function with `totalmass(M)`. `normalize(M)` divides each weight by this finite total. Both the weights and the normalizing constant depend on `K`.

D Mass of an individual state 24–27

```
p_pp = densityof(P, pp)
p_pm = densityof(P, pm)
p_mp = densityof(P, mp)
p_mm = densityof(P, mm)
```

On this discrete support, `densityof(P, pair)` gives the probability of one state. Sum the masses of the two aligned states for the probability of equal spins.

Check the partition function

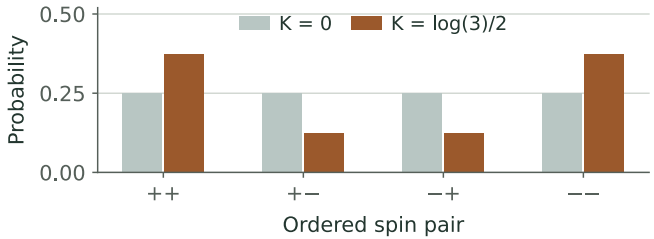
Direct enumeration gives $Z = 2e^{K} + 2e^{-K} = 4\cosh K$. At the stated coupling it equals $\frac{8}{\sqrt{3}}$. The four probabilities sum to one.

Derivatives of the log partition function

The first derivative with respect to K is the mean bond product $\tanh K$. The second is its variance $1 - \tanh^2 K$, which is $\frac{3}{4}$ at this coupling.

STATE PROBABILITIES AND DEGENERACY

Four states at two temperatures



The four pairs preserve the order of the spins. At the positive coupling in the listing, each aligned state is three times as likely as either opposed state.

Check symmetry

Simultaneously reversing both spins preserves the energy. Hence `p_pp = p_mm` and `p_pm = p_mp`. Each individual spin still has equal probabilities of +1 and −1.

Microstates per energy level

At $K = \frac{\log(3)}{2}$, summing the two probabilities in each energy level gives its mass.

Spins	Energy	Per state	Level mass
Aligned	$-J$	$\frac{3}{8}$	$\frac{3}{4}$
Opposed	$+J$	$\frac{1}{8}$	$\frac{1}{4}$

Mean bond energy

Weight each energy by its level mass to obtain $(-J)\left(\frac{3}{4}\right) + J\left(\frac{1}{4}\right) = -\frac{J}{2}$. At any coupling the mean is $-J\tanh K$, since $E = -J s_1 s_2$.

External magnetic field

```
weight(s, h) = exp(K * s[1] * s[2]
    + h * sum(s))
```

Include the dimensionless field `h` in each Boltzmann weight. Use the same four atoms and normalize the resulting measure.

Jessica Cox · bcox@mpp.mpg.de